

GROUPS WITH GRAPHICAL $C(6)$ AND $C(7)$ SMALL CANCELLATION PRESENTATIONS

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ABSTRACT. We extend fundamental results of small cancellation theory to groups whose presentations satisfy the generalizations of the classical $C(6)$ and $C(7)$ conditions in *graphical* small cancellation theory. Using these graphical small cancellation conditions, we construct lacunary hyperbolic groups and groups that coarsely contain prescribed infinite sequences of finite graphs. We prove that groups given by (possibly infinite) graphical $C(7)$ presentations contain non-abelian free subgroups.

1. INTRODUCTION

In his influential paper “Random walk in random groups” [Gr03], Gromov introduced small cancellation theory for labelled graphs as a far-reaching generalization of classical small cancellation theory. A main feature of this theory is that it allows constructions of finitely generated groups that contain prescribed subgraphs in an appropriate metric sense. Gromov applied this observation to construct *Gromov’s monster*, a group that coarsely contains an expander graph. (For details, see [AD08].)

Subsequently, Ollivier used this graphical small cancellation theory to give details on another theorem of Gromov that extends results of classical $C'(\frac{1}{6})$ small cancellation theory to its analogue in graphical small cancellation theory [Ol06].

In the present paper, we generalize the proof given by Ollivier to the graphical analogues of the classical $C(6)$ and $C(7)$ conditions. The method of proof we present is purely combinatorial; most of our statements rely solely on Lyndon’s curvature formulas [LS77], which are consequences of the formula for the Euler characteristic of planar 2-complexes, and their applications to suitable van Kampen diagrams.

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We first use our method to prove generalizations of fundamental results about classical $C(6)$ and $C(7)$ groups concerning Dehn functions and asphericity. These results have been stated before (see [Gr03] and, subsequently, [AD08, Ol06]), but proofs have never been published.

We then establish new results about groups given by non-metric graphical small cancellation presentations: Any group given by a (possibly infinite) graphical $C(7)$ presentation (see Definition 1.2) contains a non-abelian free subgroup (see Theorem 3.1). This in particular implies non-amenability and exponential growth of the group.

We moreover prove that the $Gr(6)$ condition (see Definition 1.3) can be used to construct groups that coarsely contain prescribed infinite sequences of finite graphs (see Theorem 4.3). In the view of applications to the Baum-Connes conjecture (see [HLS02]), this is of particular interest when considering an expander graph: If an expander graph $X := (X_n)_{n \in \mathbb{N}}$ admits a labelling such that the disjoint union $\sqcup_{n \in \mathbb{N}} X_n$ satisfies the $Gr(6)$ condition, then the group defined by the labelled graph X coarsely contains X . Thus, such a labelling would yield a new construction of Gromov's monster.

Continuing our investigations of groups defined by infinite sequences of finite graphs, we show that if the $Gr(7)$ condition (see Definition 1.3) is satisfied, lacunary hyperbolic groups can be constructed by passing to appropriate subsequences of defining graphs (see Proposition 5.4). If, moreover, the metric $Gr'(\frac{1}{6})$ condition is satisfied, we are able to give combinatorial conditions on the graphs that yield lacunary hyperbolicity (see Proposition 5.6).

The construction of Gromov's monster inspired further constructions of groups with extreme properties through graphical small cancellation theory (see [AD08, OW07, Si03]). Common features of all these constructions are their use of *metric* graphical small cancellation conditions and of *random* labellings of graphs. Therefore, the group presentations obtained in this manner are non-explicit. Turning them into explicit constructions has been unattainable so far.

We believe that the weaker *non-metric* small cancellation conditions we present are more accessible to making explicit constructions. Our theorems show that, in some respects, they yield results as good as the metric ones.

1.1. Graphical small cancellation conditions. Before stating the graphical small cancellation conditions, we explain how a group is constructed from a labelled graph. This idea first appeared in [RS87]. Throughout this paper, S denotes a finite set (our *alphabet*).

Let Γ be a graph. A *labelling* of Γ with letters from S is a choice of orientation on each edge of Γ and a map assigning to each edge of Γ an element of S (the *label*). A map of labelled graphs is a simplicial map that preserves the labelling. A labelling is called *reduced* if at every vertex v , any two oriented edges starting at v have distinct labels and any two oriented edges ending at v have distinct labels.

Let $M(S)$ denote the free monoid on $S \sqcup S^{-1}$. The labelling of Γ induces a map

$$\ell : \{\text{paths on } \Gamma\} \rightarrow M(S),$$

given as follows: For a path p , $\ell(p)$ is obtained by reading the labels of the edges in p starting from the initial vertex of p , and each letter is given exponent 1 if the corresponding edge is traversed in its direction and exponent -1 if it is traversed in the opposite direction.

Assume Γ is connected and fix a base vertex v in Γ . Then ℓ induces a homomorphism:

$$\ell_* : \pi_1(\Gamma, v) \rightarrow F(S),$$

where $F(S)$ denotes the free group on S . Similarly, if Γ is the disjoint union of a set of connected components with base points $\{(\Gamma_i, v_i) | i \in I\}$, we obtain a homomorphism $\ell_* : *_{i \in I} \pi_1(\Gamma_i, v_i) \rightarrow F(S)$.

Definition 1.1. For a labelled graph Γ , we define

$$G(\Gamma) := F(S) / \langle\langle \text{im } \ell_* \rangle\rangle,$$

the *group defined by* Γ . (Here $\langle\langle - \rangle\rangle$ denotes the normal subgroup generated by $-$.) Thus $G(\Gamma)$ is given by the presentation $\langle S | R \rangle$, where R is the set of words read on closed paths in Γ .

We now state the graphical small cancellation conditions. The graphical $C(n)$ and $C'(\lambda)$ conditions are based on [Ol06] and [AD08, Section 2.2], who both slightly modified Gromov's original definition [Gr03, Section 2.1]. Gromov's graphical small cancellation conditions will be denoted by $Gr(n)$ and $Gr'(\lambda)$. We call a closed path on a graph *non-trivial* if it is not 0-homotopic.

Definition 1.2. Let Γ be a labelled graph. A *piece* on Γ is a labelled path p (considered as a labelled graph) for which there are two distinct maps of labelled graphs $p \rightarrow \Gamma$. Let $n \in \mathbb{N}$ and $\lambda > 0$.

- Γ satisfies the $C(n)$ *condition* if the labelling is reduced and no nontrivial closed path is the concatenation of fewer than n pieces.
- Γ satisfies the $C'(\lambda)$ *condition* if the labelling is reduced and for any simple closed path γ , any piece p that is a subpath of Γ satisfies $|p| < \lambda|\gamma|$.

Definition 1.3. Two maps of labelled graphs $\phi_1, \phi_2 : \Gamma' \rightarrow \Gamma$ are *essentially equal* if there is an automorphism of labelled graphs $\psi : \Gamma \rightarrow \Gamma$ with $\phi_2 = \psi \circ \phi_1$. Otherwise they are *essentially distinct*. An *essential piece* on Γ is a labelled path p for which there exist two essentially distinct maps of labelled graphs $p \rightarrow \Gamma$.

- Γ satisfies the *Gr(n) condition* if the labelling is reduced and no nontrivial closed path is the concatenation of fewer than n essential pieces.
- Γ satisfies the *Gr'(λ) condition* if the labelling is reduced and for any simple closed path γ , any essential piece p that is a subpath of Γ satisfies $|p| < \lambda|\gamma|$.

Remark. The graphical small cancellation conditions are generalizations of the classical small cancellation conditions: For any classical $C(n)$ or $C'(\lambda)$ presentation $\langle S|R \rangle$, we can construct a graph as the disjoint union of cycle graphs, each labelled by (the class of cyclic conjugates and their inverses) of a relator of the presentation. If $\langle S|R \rangle$ satisfies the classical $C(n)$ or $C'(\lambda)$ condition, the constructed graph satisfies the *Gr(n)* or *Gr'(λ)* condition, respectively. If no relator is a proper power, the graph satisfies the $C(n)$ or $C'(\lambda)$ condition, respectively.

1.2. Statement of results. We first state those results which have been expected based on [Gr03], [Ol06] and [AD08]. They are generalizations of fundamental facts about classical $C(6)$ and $C(7)$ presentations. Let S be a finite set.

Theorem (cf. Theorem 2.16). *Let Γ be a $Gr(7)$ -labelled graph. Let R be the set of words read on all simple cycles of Γ . Then the presentation $\langle S|R \rangle$ satisfies the linear isoperimetric inequality:*

$$\text{Area}_R(w) \leq 8|w|.$$

In particular, if Γ is finite, then the group $G(\Gamma)$ is Gromov-hyperbolic.

Theorem (cf. Theorem 2.17). *Let Γ be a $Gr(6)$ -labelled graph. Let R be the set of words read on all simple cycles of Γ . Then the presentation $\langle S|R \rangle$ satisfies the quadratic isoperimetric inequality:*

$$\text{Area}_R(w) \leq 3|w|^2.$$

Proposition (cf. Proposition 2.18). *Let Γ be a $C(6)$ -labelled graph. Let R be the set of cyclic reductions of words read on free generating sets of the fundamental groups of the connected components of Γ . Then the presentation complex associated to $\langle S|R \rangle$ is aspherical. The group $G(\Gamma)$ defined by Γ has an at most 2-dimensional $K(G(\Gamma), 1)$ space and*

hence cohomological dimension at most 2. Therefore, $G(\Gamma)$ is torsion-free.

We next state our original results. The following theorem in particular shows that only non-amenable groups arise from graphical $C(7)$ presentations. While for finitely presented graphical $C(7)$ groups the theorem follows from the fact that non-elementary hyperbolic groups have non-abelian free subgroups (cf. [Ko98, Theorem 5.1]), it is new for infinitely presented $C(7)$ groups.

Theorem (cf. Theorem 3.1). *Let Γ be a $C(7)$ -labelled graph. Then $G(\Gamma)$ contains a non-abelian free subgroup unless it is trivial or infinite cyclic.*

In fact, we explain in Remark 3.3 that $G(\Gamma)$ is trivial or infinite cyclic if and only if some easily checkable conditions on the graph hold.

We next consider how the defining graph Γ of a group embeds into the Cayley graph $\text{Cay}(G(\Gamma), S)$ as a metric space. Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of connected finite graphs and let $\Gamma := \bigsqcup_{n \in \mathbb{N}} \Gamma_n$ be their disjoint union. We endow Γ with a metric that coincides with the graph metric on each connected component such that $d(\Gamma_{a_n}, \Gamma_{b_n}) \rightarrow \infty$ as $a_n + b_n \rightarrow \infty$ assuming $a_n \neq b_n$ for almost all n . We call the resulting metric space the *coarse union* of the Γ_n . A *coarse embedding* is a map $f : X \rightarrow Y$, where X and Y are metric spaces, such that for all sequences $(x_n, y_n)_{n \in \mathbb{N}}$ in $X \times X$ we have $d(x_n, y_n) \rightarrow \infty \Leftrightarrow d(f(x_n), f(y_n)) \rightarrow \infty$.

The following theorem shows that, whenever we find a labelling of an infinite sequence of finite graphs such that their disjoint union satisfies the $Gr(6)$ condition, we can construct a finitely generated group that coarsely contains the coarse union of these graphs:

Theorem (cf. Theorem 4.3). *Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of connected finite graphs such that $\Gamma := \bigsqcup_{n \in \mathbb{N}} \Gamma_n$ is $Gr(6)$ -labelled and such that $|\Gamma_n|$ is unbounded. Then the coarse union Γ embeds coarsely into $\text{Cay}(G(\Gamma), S)$.*

This map is also injective as stated by Gromov [Gr03] and proven in Lemma 4.1.

A finitely generated group is called *lacunary hyperbolic* if one of its asymptotic cones is an \mathbb{R} -tree. The following proposition shows that the $Gr(7)$ condition can be used to construct lacunary hyperbolic groups:

Proposition (cf. Proposition 5.4). *Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of finite, connected labelled graphs such that their disjoint union is $Gr(7)$ -labelled. Then there exists an infinite subsequence of graphs $(\Gamma_{k_n})_{n \in \mathbb{N}}$ such that $G(\bigsqcup_{n \in \mathbb{N}} \Gamma_{k_n})$ is lacunary hyperbolic.*

Moreover, we apply results of [Ol06] and an argument given in [OOS09, Proposition 3.12], where the question when a group given by an infinite classical $C'(\frac{1}{6})$ presentation is lacunary hyperbolic is answered, to show:

Proposition (cf. Proposition 5.6). *Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of finite, connected graphs such that*

- $\Gamma := \sqcup_{n \in \mathbb{N}} \Gamma_n$ has a $Gr'(\frac{1}{6})$ -labelling and
- $\Delta(\Gamma_n) = O(g(\Gamma_n))$,

where $\Delta(\Gamma_n)$ denotes the diameter and $g(\Gamma_n)$ denotes the girth of each graph. (If Γ_n is a tree, set $g(\Gamma_n) = 0$.) Then $G(\Gamma)$ is lacunary hyperbolic if and only if the set of girths $L := \{g(\Gamma_n) | n \in \mathbb{N}\}$ is sparse, i.e. for all $K \in \mathbb{R}^+$ there exists $a \in \mathbb{R}^+$ such that $[a, aK] \cap L = \emptyset$.

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2. EXTENDING CLASSICAL SMALL CANCELLATION THEORY

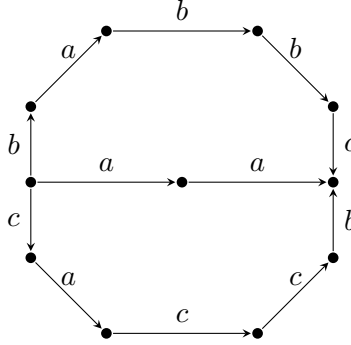
2.1. Examples and first observations. Following the definitions of the graphical small cancellation conditions given in the introduction, we provide two examples and make some basic observations about pieces and the graphical small cancellation conditions.

Example 2.1. Let $S = \{a, b, c\}$, and let Γ be as in Figure 1. The group $G(\Gamma)$ is given by the presentation

$$\langle a, b, c | a^2 c^{-1} b^{-2} a^{-1} b^{-1}, a^2 b^{-1} c^{-2} a^{-1} c^{-1} \rangle.$$

The graph Γ satisfies the $C'(\frac{1}{6})$ condition: The labelling is reduced, any piece that is subpath of a simple cycle has length at most 1, and any cycle has length at least 7. Note that the presentation does not satisfy the classical $C'(\frac{1}{6})$ condition. Also note that this is a maximal graphical $C'(\frac{1}{6})$ presentation in the alphabet S , i.e. one cannot add more S -labelled cycles to the graph retaining the $C'(\frac{1}{6})$ condition, since any such cycle would produce new pieces of length 2 on Γ that would lead to a violation of the $C'(\frac{1}{6})$ condition.

Example 2.2. Let G be a group generated by a finite set S . Then the Cayley graph $\text{Cay}(G, S) =: \Gamma$ has a natural reduced labelling, and we have $G = G(\Gamma)$. Note that a path on a graph with a reduced labelling is uniquely determined by its starting vertex and its label. The group


 FIGURE 1. The $C'(\frac{1}{6})$ -labelled graph Γ .

of labelled-graph-automorphisms of Γ acts transitively on the vertex set of Γ . Therefore, whenever a labelled path has two maps to Γ , they are essentially equal. Thus Γ has no essential pieces and satisfies any $Gr(n)$ or $Gr(\lambda)$ condition. Unless the group is free on S , or trivial, it will not satisfy $C(2)$ or $C'(1)$, i.e. no nontrivial $Gr(n)$ or $Gr'(\lambda)$ condition, since there will be two distinct cycles with the same label and thus a cycle made up of one piece.

Remark. Any essential piece is a piece. Thus, the $C(n)$ condition implies the $Gr(n)$ condition, and the $C'(\lambda)$ condition implies the $Gr'(\lambda)$ condition. The treatment of the conditions from Definition 1.2 and those from Definition 1.3 is very similar; in fact all results of Subsections 2.1-2.3 hold for both conditions and for pieces and essential pieces alike. We will therefore mostly restrict ourselves to stating them for the $C(n)$ or $C'(\lambda)$ conditions and for pieces only. The only difference in statements and proofs is that “piece” has to be replaced by “essential piece”, “unique” by “essentially unique”, “equal” by “essentially equal”, etc.

Backtracking in a path on a graph Γ means that an edge of Γ is traversed two consecutive times in opposite directions. A path in a graph is *reduced* if there is no backtracking. The reduction of a path is obtained by removing any backtracking. We have the following observations for a piece p :

- The reduction of p is a piece.
- Every subpath of p is a piece.
- The inverse path of p is a piece.

Lemma 2.3. *Let Γ be a labelled graph, $n \in \mathbb{N}$ and $\lambda > 0$. Then the following are equivalent:*

- i) Γ satisfies $C(n)$.

- ii) *No reduced cycle is the concatenation of fewer than n pieces.*
- iii) *No simple cycle is the concatenation of fewer than n pieces.*

Moreover, if Γ satisfies $C'(\lambda)$, then Γ satisfies $C(\lfloor \frac{1}{\lambda} \rfloor + 1)$.

Proof. The implications i) \Rightarrow ii) \Rightarrow iii) are obvious. Now suppose iii) holds. Let γ be a nontrivial cycle on Γ , and suppose $\gamma = p_1 p_2 \dots p_k$ where p_i is a piece for each i . Then the reduction of γ may be written as $q_1 q_2 \dots q_l$ where each q_i is a subpath of the reduction of a p_j , and $l \leq k$. Let γ' be a subpath of γ that is a simple cycle. Then $\gamma' = r_1 r_2 \dots r_m$, where each r_i is a subpath of some q_j , and $m \leq l$. By assumption, we have $m \geq n$ and hence $k \geq l \geq m \geq n$.

The last statement follows from the implication iii) \Rightarrow i). \square

2.2. Diagrams. We briefly recall tools and results of classical small cancellation theory, which we will use to present graphical small cancellation theory. For an introduction to classical small cancellation theory see [LS77, Chapter V].

A *singular disk diagram* D in the alphabet S is a finite, simply connected, 2-dimensional CW-complex embedded into the plane with the following additional data:

- Each 1-cell is oriented and labelled by an element of S .
- A base vertex v in the topological boundary of $\mathbb{R}^2 \setminus D$ is specified. (We also denote (D, v) . We omit mentioning the base vertex if this is not necessary.)

A *simple disk diagram* is a singular disk diagram that is homeomorphic to a disk. Equivalently, a simple disk diagram is singular disk diagram without cut-points.

The *boundary path* ∂D of a singular disk diagram D is the simplicial path obtained by traversing the topological boundary of $\mathbb{R}^2 \setminus D$ from the base vertex in counterclockwise direction. The *label* of ∂D (or *boundary label* of D) is the element of the free monoid on $S \sqcup S^{-1}$ obtained by reading the labels of the edges of ∂D , where each letter is given exponent 1 if the corresponding edge is traversed in its direction and exponent -1 if it is traversed in the opposite direction.

We also introduce spherical diagrams following [CH82]. A spherical complex is a set of 2-spheres embedded into \mathbb{R}^3 connected by simple curves such that no sphere contains the other, such that the intersection of two spheres consists of at most one point and such that the whole complex is simply connected. (Recall that a curve is simple if it has no self-intersections.)

A *spherical diagram* in the alphabet S is finite 2-complex tessellating a spherical complex, where each 1-cell is oriented and labelled by an element of S . A *simple* spherical diagram is a spherical diagram tessellating a single 2-sphere. It is our convention that a spherical diagram has empty boundary.

The word “diagram” will refer to a singular disk diagram or a spherical diagram. The *area* of a diagram is the number of its 2-cells (or *faces*).

For the finite set S , let $M(S)$ denote the free monoid on $S \sqcup S^{-1}$. For a set of words $R \subset M(S)$, a diagram over R is a diagram D such that for each face f of D there exist an orientation and a vertex $v \in \partial f$ such that the boundary label of f read from v lies in R . Given a labelled graph Γ , a *diagram over Γ* is a diagram over the set of words read on nontrivial cycles on Γ .

Let $w \in M(S)$. A *diagram for w over R* (respectively *over Γ*) is a diagram over R (respectively Γ) whose boundary label is w . A *minimal diagram for w over R* (respectively Γ) is a diagram D for w over R (respectively Γ) such that

- The area of D is minimal among all diagrams for w over R (respectively Γ), and
- the number of edges of D is minimal among all such diagrams of minimal area.

The following theorem, a version of the so-called “van Kampen Lemma”, is central in small cancellation theory. Its proof can be found in [LS77, Section V.1].

Theorem 2.4. *Let G be a group given by the presentation $\langle S | R \rangle$. An element w of the free monoid $M(S)$ over S satisfies $w = 1$ in G if and only if there exists a singular disk diagram over R such that the boundary label of D is w .*

An *arc* in a diagram is a path all interior vertices of which have degree 2 and whose initial and terminal vertices have degrees different from 2. A *spur* is an arc which has a vertex of degree 1. A face f of a diagram D is called *interior* if the intersection $\partial f \cap \partial D$ contains no edge. All other faces are called *exterior* or *boundary faces*.

Let p, q be positive integers. A diagram D is a (p, q) -*diagram* if every interior vertex has degree at least p and if the boundary of every interior face consists of at least q edges. A diagram is a $[p, q]$ -*diagram* if it is a (p, q) -diagram and if the boundary of every face consists of at least q edges. Note that a spherical (p, q) -diagram is a $[p, q]$ -diagram.

If D is a diagram in which the boundary path of each interior face is the concatenation of at least q arcs, we can consider D as a $(3, q)$ -diagram if we “forget” vertices of degree two. Forgetting a vertex v of degree two means removing the vertex and replacing the two edges incident at v by a single edge (cf. Figure 2). In this context, labels and orientations of edges will play no role and will therefore be ignored.



FIGURE 2. Forgetting a vertex of degree two.

The following are standard results from classical small cancellation theory which we will use later on:

Theorem 2.5 (Curvature formula I, cf. [LS77, Corollary V.3.4]). *Let D be a $(3, 6)$ -singular disk diagram with at least two faces. For a face f , let $i(f)$ denote the number of its interior edges. Then*

$$\sum_{f \text{ a boundary face of } D} (4 - i(f)) \geq 6.$$

In fact, more refined versions of this result can be found in [MW02] and [St90].

Theorem 2.6 (Curvature formula II, cf. [LS77, Corollary V.3.3]). *Let D be a $[3, 6]$ -singular disk diagram. For a vertex v , let $d(v)$ denote its degree. Then*

$$\sum_{v \in \partial D} (2 + \frac{1}{2} - d(v)) \geq 3,$$

where the sum is taken over all vertices in the boundary.

The following theorem is an immediate conclusion of [LS77, Theorem 6.2] and the first curvature formula.

Theorem 2.7 (Area theorem I). *Let D be a $(3, 6)$ -singular disk diagram. Then*

$$|D| \leq 3|\partial D|^2,$$

where $|D|$ is the number of faces of D and $|\partial D|$ is the number of its boundary edges.

A stronger statement holds for $(3, 7)$ -diagrams. It is proved in [St90, Proposition 2.7].

Theorem 2.8 (Area theorem II). *Let D be a $(3, 7)$ -singular disk diagram. Then*

$$|D| \leq 8|\partial D|.$$

2.3. Basic tools and results. We will now explain how to apply the aforementioned results from classical small cancellation theory to graphical small cancellation presentations. The following definition is given in [Ol06].

Definition 2.9 (To originate from Γ). Let Γ be a $C(2)$ -labelled graph, and let R be the set of words read on a set of nontrivial cycles on Γ . We identify each $r \in R$ with the labelled graph that is a cycle of length $|r|$ and bears the label r . With this identification, r *lifts* (i.e. maps) to Γ and, since $C(2)$ is satisfied, in a unique way. Hence, for a diagram over R , the boundary of each face lifts uniquely to Γ . Let p be a subpath of an interior arc of a diagram D lying in the boundary of faces f_1 and f_2 . There are lifts $\partial f_1 \rightarrow \Gamma$ and $\partial f_2 \rightarrow \Gamma$. If the image of p in Γ coincides for both lifts, then we say p *originates from* Γ .

Remark. Any interior arc of a diagram that does not originate from Γ is a piece.

Remark. The above definition can be extended to $Gr(2)$ -labelled graphs. In that case, lifts are considered up to composition with an automorphism of Γ . Any interior arc of a diagram that does not essentially originate from Γ is an essential piece. As stated before, all proofs and conclusions of Subsections 2.1-2.3 for $C(n)$ - (respectively $C'(n)$ -) labelled graphs are true for $Gr(n)$ - (respectively $Gr'(\lambda)$ -) labelled graphs as well. We will not state them doubly to avoid confusing notation.

The following lemma is our main tool in extending classical small cancellation theory to graphical small cancellation theory. It is based on [Ol06], in which the $C'(\frac{1}{6})$ condition is investigated.

Lemma 2.10. *Let D be a singular disk diagram over a $C(n)$ -labelled graph Γ , where $n \geq 6$. Then D has a subdiagram that is a simple disk diagram with at least one face, with freely trivial boundary word and all interior edges of which originate from Γ , or removing all edges of D originating from Γ and merging the corresponding 2-cells yields a $(3, n)$ -diagram D' over Γ , all faces of which are simply connected.*

Proof. Suppose D does not have a simple subdiagram with freely trivial boundary word all interior edges of which originate from Γ . Let D' denote the diagram with possible “faces with holes” obtained by

removing all edges originating from Γ . Note that all interior arcs of D' are pieces. We show that D' is in fact a diagram and all faces are simply connected. The small cancellation assumption then yields that D' is a $(3, n)$ -diagram.

Suppose in D' there is a face f with $k > 0$ holes, i.e. f is homotopic to a 2-disk with k holes, and these holes are filled by (the interiors of) singular disk diagrams. Let Δ be such a singular disk diagram. Then Δ has at most one vertex in common with the exterior boundary of f . We may assume that each face of Δ is simply connected. By construction, there are no spurs in Δ , as a spur would consist of edges originating from Γ . By assumption, all faces of Δ bear the labels of nontrivial cycles in Γ . Since all arcs of Δ are pieces, Δ is a $[3, 6]$ -diagram where all but at most one vertex have degree ≥ 3 , contradicting the second curvature formula. \square

Suppose there is a subdiagram with at least one face that has freely trivial boundary word. Then we can fold its boundary and remove the subdiagram without altering anything else in the diagram as illustrated in Figure 3. The resulting diagram has smaller area than the original one. Hence, removing all edges originating from Γ on a diagram D of minimal area for a word w yields a diagram as in the second case of the Lemma. We therefore obtain:

Corollary 2.11. *Let $w \in M(S)$ satisfying $w = 1$ in $G(\Gamma)$, and let D be a minimal diagram for w over Γ . Then no interior edge of D originates from Γ , and therefore, D is a $(3, n)$ -diagram, all faces of which are simply connected.*

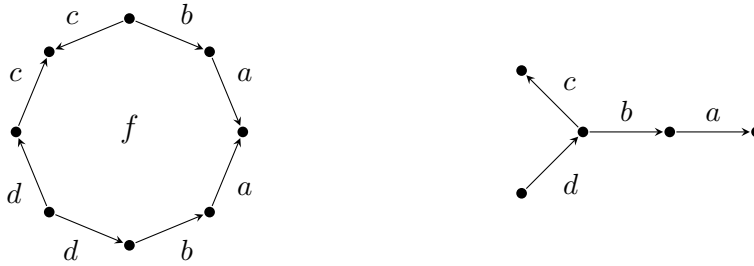


FIGURE 3. Folding a face with freely trivial boundary word.

The following lemma is proved just as Lemma 2.10 with the additional observation that there is no spherical $[3, 6]$ -diagram with more than one face. This follows from the second curvature formula.

Lemma 2.12. *Let D be a simple spherical diagram over a $C(6)$ -labelled graph Γ . Then D has a subdiagram that is a simple disk diagram with at least one face, with freely trivial boundary word and all interior edges of which originate from Γ , or all edges of D originate from Γ .*

The set of all words read on nontrivial cycles of a labelled graph Γ is generally infinite. The set of all words read on all *simple* cycles of Γ generates the same normal subgroup of $F(S)$. If Γ has finitely generated fundamental group, this set is finite. The following lemma improves the conclusions of Lemma 2.10 to show, that considering the presentation given by the words read on all simple cycles is sufficient to obtain $(3, n)$ -diagrams from $C(n)$ labelled graphs if $n \geq 6$.

Lemma 2.13. *Let Γ be a $C(n)$ labelled graph for $n \geq 6$. Let $w \in M(S)$ satisfying $w = 1$ in $G(\Gamma)$. Then there exists a diagram D for w such that no interior edge of D originates from Γ and such that every face bears the label of a simple cycle in Γ .*

Note that by Lemma 2.10, D is a $(3, n)$ diagram, all faces of which are simply connected.

Proof of the lemma. Let D be a diagram for w over Γ with a minimal number of edges whose number of vertices is minimal among all diagrams for w over Γ with a minimal number of edges.

Suppose D has an interior edge originating from Γ between faces f_1 and f_2 . Then this edge can be removed, and a new face f is obtained. If f has a freely nontrivial boundary word, the resulting diagram is a diagram over Γ contradicting the minimality assumptions on D . If f has a freely trivial boundary word, it can be folded and removed to leave a diagram over Γ , again contradicting the minimality assumptions. Thus no interior edge originates from Γ . In particular, every face is simply connected. Also note that there is no spur with an endpoint in the interior of D , as such a spur would lie entirely in one face and consist of edges originating from Γ .

Suppose D has a face f bearing a boundary word that cannot be read on a simple cycle. Since f is simply connected and since there are no spurs inside f , there are two *distinct* vertices v_1 and v_2 on ∂f lifting to the same point in Γ via the lift $\partial f \rightarrow \Gamma$. We can identify v_1 and v_2 , i.e. “pinch” v_1 and v_2 together to a single vertex (cf. Figure 4). If a face with freely trivial boundary word arises, it can be folded and removed to leave a diagram over Γ . This contradicts the minimality of number of edges of D . If no face with freely trivial boundary arises, the diagram obtained by pinching vertices is a diagram over Γ with the

same number of edges as D , but with fewer vertices, again contradicting the assumptions. \square

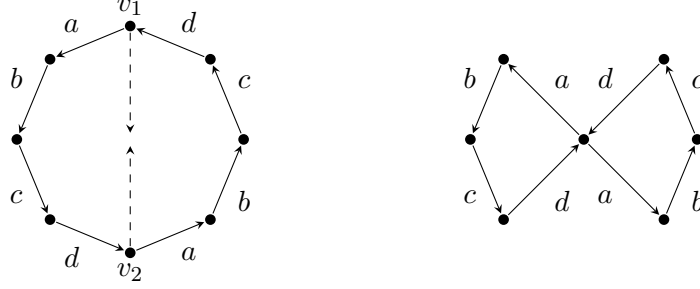


FIGURE 4. Pinching together two vertices lying on a face that has a boundary path not lifting to a simple cycle.

Definition 2.14 (Gromov-hyperbolic group). Let X be a geodesic metric space. A *geodesic triangle* in X is a triple of geodesics (g_1, g_2, g_3) such that the endpoint $\tau(g_1)$ of g_1 equals the starting point $\iota(g_2)$ of g_2 , $\tau(g_2) = \iota(g_3)$ and $\tau(g_3) = \iota(g_1)$. Let $\delta \geq 0$. Then X is called δ -*hyperbolic* if and only if for all geodesic triangles (g_1, g_2, g_3) , g_3 is contained in the δ -neighborhood of $g_1 \cup g_2$. The space X is called *hyperbolic* if it is δ -hyperbolic for some $\delta \geq 0$.

Let G be a group generated by a finite set S . The Cayley graph $\text{Cay}(G, S)$ of G with respect to S is a connected graph endowed with the graph metric on its vertex set. If we consider each edge of $\text{Cay}(G, S)$ isometric to the unit interval $[0, 1]$, $\text{Cay}(G, S)$ becomes a geodesic metric space. We say G is δ -hyperbolic with respect to S if $\text{Cay}(G, S)$ is δ -hyperbolic. The group G is called *Gromov-hyperbolic* if it is δ -hyperbolic with respect to S for some $\delta \geq 0$ and some finite generating set S .

Definition 2.15 (Isoperimetric inequality). Let G be a group given by the presentation $\langle S | R \rangle$. For $w \in M(S)$ satisfying $w = 1$ in G , let $\text{Area}_R(w)$ denote the minimal area of a singular disk diagram for w over R . The *Dehn function* of G with respect to $\langle S | R \rangle$ is the map $f : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(l) = \max\{\text{Area}_R(w) \mid |w| = l\}.$$

We say a group presentation satisfies a *linear* (resp. *quadratic*) *isoperimetric inequality* if the corresponding Dehn function is bounded from above by a linear (resp. quadratic) map $\mathbb{R} \rightarrow \mathbb{R}$.

We state two immediate consequences of Lemma 2.13 using the area theorems. For classical $C(6)$ and $C(7)$ small cancellation presentations, these are well-known. It is shown in [Al91, Theorem 2.5 and Proposition 2.10] that a group is Gromov-hyperbolic if and only if it has a finite presentation satisfying a linear isoperimetric inequality.

We choose to state the theorems for the $Gr(7)$ and $Gr(6)$ conditions (instead of the graphical $C(7)$ and $C(6)$ conditions) as they are more general.

Theorem 2.16. *Let Γ be a $Gr(7)$ -labelled graph. Let R be the set of words read on all simple cycles of Γ . Then the presentation $\langle S|R \rangle$ satisfies the linear isoperimetric inequality:*

$$\text{Area}_R(w) \leq 8|w|.$$

In particular, if Γ is finite, then the group $G(\Gamma)$ is Gromov-hyperbolic.

Theorem 2.17. *Let Γ be a $Gr(6)$ -labelled graph. Let R be the set of words read on all simple cycles of Γ . Then the presentation $\langle S|R \rangle$ satisfies the quadratic isoperimetric inequality:*

$$\text{Area}_R(w) \leq 3|w|^2.$$

2.4. Asphericity. For a group presentation $\langle S|R \rangle$, the *presentation complex* is a CW-complex constructed as follows: The 0-skeleton is a single point v . For each generator in S , a labelled, oriented loop is glued to v , and for each relator r in R , a 2-cell whose boundary label is r is glued along its boundary label onto the 1-skeleton. A CW-complex is called *aspherical* if its universal cover is contractible.

We show that a group given by a graphical $C(6)$ presentation has an aspherical presentation complex. This is a generalization of the corresponding well-known result for classical $C(6)$ presentations where no relator is a proper power. It can easily be seen from Example 2.2 that this result does not hold for $Gr(6)$ presentations.

Proposition 2.18. *Let Γ be a $C(6)$ -labelled graph. Let R be the set of cyclic reductions of words read on free generating sets of the fundamental groups of the connected components of Γ . Then the presentation complex associated to $\langle S|R \rangle$ is aspherical.*

Together with [Br81, VIII.2] this yields:

Corollary 2.19. *The group $G(\Gamma)$ defined by Γ has an at most 2-dimensional $K(G(\Gamma), 1)$ space and hence cohomological dimension at most 2. Therefore, $G(\Gamma)$ is torsion-free.*

Lemma 2.20. *Let Γ and R be as in Proposition 2.18. Let R' be a proper subset of R . Then the homomorphism $\langle S|R' \rangle \rightarrow \langle S|R \rangle$ induced by the identity on S is not injective. In particular, if R is infinite, then $G(\Gamma)$ is not finitely presented.*

We postpone the proof of this lemma as it will be derived from our proof of Proposition 2.18. To prove asphericity, we will show that spherical diagrams over a $C(6)$ graph are reducible in an appropriate sense. We use definitions and results from [CH82] that allow an algebraic treatment of diagrams:

Let $R \subset F(S)$, and set $R^S := \{gr^\epsilon g^{-1} | r \in R, g \in F(S), \epsilon \in \{\pm 1\}\} \subset F(S)$. A *sequence over R* is a finite sequence of elements of R^S . On all sequences over R , we consider the following operations:

- Exchange: Replace a pair (x, y) by (xyx^{-1}, x) or by $(y, y^{-1}xy)$.
- Deletion: Delete a pair (x, x^{-1}) .
- Insertion: Insert at any position a pair (x, x^{-1}) for any $x \in R^S$.

We call two sequences over R *equivalent* if one can be transformed into the other by a finite sequence of these operations. We call any finite sequence over R an *identity sequence* if the product of its elements (taken in the order they appear in the sequence) is trivial in $F(S)$. We call a sequence *trivial* if it is equivalent to the empty sequence.

To a diagram (D, v) over R , we can associate a sequence Σ over R as follows: From D , we construct a singular disk diagram D' by “ungluing” faces of D along edges, such that D' is a bouquet of faces, each connected to the basepoint v by an arc, such that the boundary word of D' is freely equal to that of D . If D is spherical, we choose some embedding in the plane after ungluing the first edge. The boundary word of D' is freely trivial in that case.

The sequence Σ is then the sequence of the labels of faces of D' , each conjugated by the label of the path that connects it to v , read in the order in which they appear in the boundary path. The resulting sequence is called a *derived sequence* for (D, v) . Its length is equal to the number of faces of D , and the product over all elements of Σ is freely equal to the boundary word of D .

Whilst a derived sequence for a diagram (D, v) is not unique, [CH82, Proposition 8] yields that any two derived sequences for a singular disk diagram (D, v) are equivalent. Moreover, [CH82, Corollary 1 of Proposition 8] implies that if Σ and Σ' are derived sequences for a spherical diagram D , Σ is trivial if and only if Σ' is.

For any sequence Σ over R , we can construct a singular disk diagram D over R which has Σ as a derived sequence. (This is done by reversing the procedure described above.) If Σ is an identity sequence, the

diagram has freely trivial boundary word. This freely trivial boundary word can be “sewn up” to obtain a spherical diagram. Sewing up means that, whenever there are two edges bearing inverse labels in the boundary, we fold them together until eventually we are left with a spherical diagram. (See Figure 3.) We call a diagram constructed from a sequence Σ an *associated diagram* for Σ .

The following theorem is deduced from [CCH81, Proposition 1.3 and Proposition 1.5]. A presentation $\langle S|R \rangle$ is *concise* if for any relator $r \in R$, no $r' \in R$ that is distinct from r is conjugate to r or r^{-1} .

Theorem 2.21. *Let $\langle S|R \rangle$ be a presentation where each relator is non-trivial and freely reduced. Then the associated presentation complex is aspherical if and only if:*

- *The presentation is concise,*
- *no relator is a proper power,*
- *and any identity sequence over R is trivial.*

We will show that the presentation in our proposition satisfies these three conditions.

Lemma 2.22. *Let Γ be a $C(2)$ labelled graph. Let R be the set of cyclic reductions of words read on free generating sets of the fundamental groups of the connected components of Γ (with respect to some base points). Then R is concise and contains no proper powers.*

Proof. For any cycle γ on Γ , there exists a maximal initial subpath p such that $\gamma = p\gamma'p^{-1}$ for some subpath γ' . Then γ' is a cycle and, since the labelling of Γ is reduced, its label $\ell(\gamma')$ equals the cyclic reduction of $\ell(\gamma)$. We call γ' the *cyclic reduction* of γ .

Let Γ_i for $i \in I$, where I is some index set, denote the connected components of Γ , and let ν_i denote a base vertex in Γ_i for each i . Let γ be the reduced representative of an element of a free generating set of some $\pi_1(\Gamma_i, \nu_i)$ and γ' its cyclic reduction such that $\gamma = p\gamma'p^{-1}$ for some path p . Suppose $\ell(\gamma') = w^n$ for $w \neq 1$ and $n > 1$. Let $\gamma' = \gamma_0 \dots \gamma_{n-1}$ such that $\ell(\gamma_j) = w$ for $j \in \mathbb{Z}/n\mathbb{Z}$.

Then the map of labelled graphs $\gamma' \rightarrow \gamma'$ induced by $\gamma_j \rightarrow \gamma_{j+1}$ must be the identity, for otherwise γ' would have two distinct maps of labelled graphs to Γ and thus would be a single piece. Therefore, $\gamma_j = \gamma_{j+1}$ for all j , and each γ_j is a cycle. Thus $[\gamma] = [p\gamma_0p^{-1}]^n$ in $\pi_1(\Gamma_i, \nu_i)$. But it is easy to see that an element of a free generating set of $\pi_1(\Gamma_i, \nu_i)$ cannot be a proper power.

Now assume that there are two distinct relators r and r' in R coming from cycles $[\gamma]$ and $[\delta]$ in the free generating sets, where γ and δ are reduced, such that r' is conjugate to r (or r^{-1}). Since r and r' are

cyclically reduced, they coincide up to a cyclic permutation (and possibly inversion). Again, consider the cyclic reductions γ' and δ' . Then we can perform a cyclic shift on δ' such that the resulting cycle $\tilde{\delta}'$ has the same label as γ' (or $(\gamma')^{-1}$). But then $\tilde{\delta}'$ must equal γ' (or $(\gamma')^{-1}$) for otherwise, it would be a single piece. This implies that $[\gamma]$ and $[\delta]$ lie on the same connected component of Γ , say Γ_i , and that they are conjugate in $\pi_1(\Gamma_i, \nu_i)$ (or conjugate up to inversion). This cannot hold for two elements of a free generating set. \square

Lemma 2.23. *Let Γ be a connected graph with a reduced labelling. Let R be the set of cyclic reductions of words read a set of free generators of $\pi_1(\Gamma, \nu)$ for some $\nu \in \Gamma$. Let (D, v) be a simple disk diagram over R whose boundary word is freely trivial and such that all interior edges of D originate from Γ . Then any derived sequence is a trivial identity sequence.*

Proof. Let $\{\phi_i | i \in I\}$ be a set of free generators of $\pi_1(\Gamma, \nu)$ such that

$$\phi_i = [\gamma_i \rho_i \gamma_i^{-1}],$$

where $\ell(\rho_i)$ is equal to the cyclic reduction of $\ell_*(\phi)$ and γ_i is a reduced path. Denote the terminal vertex of γ_i by ν_i . Then $\{\ell(\rho_i) | i \in I\} = R$, and $\{\ell_*(\phi_i) | i \in I\} \subset R^S$.

The diagram D globally lifts to Γ via a map $\lambda : D^1 \rightarrow \Gamma$, where D^1 denotes the 1-skeleton of D . Let $v \in \partial D$. Since Γ is connected, we may assume that ν was chosen such that $\nu = \lambda(v)$. The path $\lambda(\partial D)$ is a cycle on Γ based at ν . Since the labelling of Γ is reduced, the assumption that the boundary label of D is freely trivial implies that $\lambda(D)$ is 0-homotopic, i.e. $[\lambda(\partial D)] = 1 \in \pi_1(\Gamma, \nu)$.

We number the faces of D as f_1, \dots, f_n and choose an orientation on each face. For a face f_j with boundary label $\ell(\rho_i)^{\epsilon_j}$, where $i \in I$ and $\epsilon_j \in \{\pm 1\}$, let v_j denote the vertex that lifts to ν_i , and set $t_j := i$. Then there are paths p_j in D^1 from v to v_j such that in $\pi_1(\Gamma, \nu)$:

$$1 = [\lambda(\partial D)] = [\lambda(p_1) \rho_{t_1}^{\epsilon_1} \lambda(p_1)^{-1}] \dots [\lambda(p_n) \rho_{t_n}^{\epsilon_n} \lambda(p_n)^{-1}].$$

For each j , the path $\lambda(p_j) \gamma_{t_j}^{-1}$ is a cycle based at ν ; we denote $\eta_j := [\lambda(p_j) \gamma_{t_j}^{-1}]$. The sequence

$$\Sigma := (\ell_*(\eta_1 \phi_{t_1}^{\epsilon_1} \eta_1^{-1}), \dots) = (\ell_*(\eta_1) \ell_*(\phi_{t_1}^{\epsilon_1}) \ell_*(\eta_1^{-1}), \dots)$$

is a derived sequence for (D, v) . For each j , we can express η_j in the free generators $\{\phi_i | i \in I\}$ as a reduced word W_j and write:

$$\Sigma = (\ell_*(W_1) \ell_*(\phi_{t_1}^{\epsilon_1}) \ell_*(W_1^{-1}), \dots, \ell_*(W_n) \ell_*(\phi_{t_n}^{\epsilon_n}) \ell_*(W_n^{-1})).$$

Now suppose $W_1 \neq 1$, and let the first letter of W_1 be $\phi_{f_1}^{e_1}$. Then we can insert the pair $(\ell_*(\phi_{f_1}^{e_1}), \ell_*(\phi_{f_1}^{-e_1}))$ into Σ at the first position and perform an exchange operation to obtain

$$(\ell_*(\phi_{f_1}^{e_1}), \ell_*(W_1')\ell_*(\phi_{t_1}^{e_1})\ell_*(W_1'^{-1}), \ell_*(\phi_{f_1}^{-e_1}), \ell_*(W_2)\ell_*(\phi_{t_2}^{e_2})\ell_*(W_2^{-1}), \dots),$$

where $\phi_{f_1}^{e_1}W_1' = W_1$, and W_1' is shorter than W_1 . Iterating this procedure and applying it to all W_i yields a sequence Σ' of the form

$$\Sigma' = (\ell_*(\phi_{g_1}^{h_1}), \dots, \ell_*(\phi_{g_N}^{h_N})),$$

where each $h_i \in \{\pm 1\}$. Since $1 = \phi_{g_1}^{h_1} \dots \phi_{g_N}^{h_N}$ in $\pi^1(\Gamma, \nu)$ and since a free reduction on the right hand side of this equation corresponds to a deletion operation in Σ' , we see that Σ' can be transformed into the empty sequence by deletion operations. \square

Proof of the proposition. Let Σ be a nontrivial identity sequence over R of minimal length. Then there is an associated spherical diagram D . We may restrict ourselves to the case that D is a simple spherical diagram, as the general case can be constructed from this. Lemma 2.12 implies that all edges of D originate from Γ , or that D has a subdiagram f that is a simple disk diagram with at least one face and with freely trivial boundary word, and all interior edges of f originate from Γ . Assume the latter.

Note that changing the base point of a diagram corresponds to a conjugation of all elements of the derived sequence with an element $g \in F(S)$. A sequence is equivalent to a shorter sequence if and only if its conjugate is. Hence we can assume that the base point v lies in the boundary of f .

We now associate to (D, v) a derived sequence Σ' that has an initial subsequence σ that is a derived sequence for (f, v) : We cut f out of D and glue f and $D \setminus f$ together at the point v such that the boundary labels of f and $D \setminus f$ read from v are inverse. Let Σ' be the concatenation of a sequence σ for f and one for $D \setminus f$. Since f has at least one face, σ is nonempty. Note that Σ and Σ' have the same length. By Lemma 2.23, σ reduces to the trivial sequence, and therefore Σ' is equivalent to a shorter sequence. Thus the minimality assumption on Σ yields that Σ' is trivial. Now, as mentioned before, [CH82, Corollary 1 to Proposition 8] implies that Σ is trivial, a contradiction.

If all edges of D originate from Γ , we can unglue two faces along any edge of D to obtain a simple disk diagram with freely trivial boundary word that globally lifts to Γ . Lemma 2.23 yields that any derived sequence is trivial, and therefore Σ is trivial. \square

Proof of Lemma 2.20. Let R be as in Proposition 2.18. Assume there is $r \in R$ such that $r \in \langle\langle R \setminus \{r\} \rangle\rangle$. Then there exists a diagram for r over $R \setminus \{r\}$ and hence a spherical diagram D over R with exactly one face labelled by r . There exists a derived sequence for D containing a conjugate of r , but no conjugate of r^{-1} , and therefore the sequence is not trivial, which is a contradiction. \square

3. FREE SUBGROUPS

We prove that graphical $C(7)$ groups are either cyclic, or they have non-abelian free subgroups.

Theorem 3.1. *Let Γ be a $C(7)$ -labelled graph. Then $G(\Gamma)$ contains a non-abelian free subgroup unless it is trivial or infinite cyclic.*

Corollary 3.2. *Graphical $C(7)$ groups have exponential growth and are non-amenable, unless they are trivial or infinite cyclic.*

Remark 3.3. We show that it is easy to check if $G(\Gamma)$ is trivial or infinite cyclic. First, either one of the curvature formulas implies that there is no diagram over Γ whose boundary has length 1 and which has more than one face. Hence, $G(\Gamma)$ is trivial if and only if it contains a loop of length 1 labelled with s for each $s \in S$.

Second, we check when an infinite cyclic group can arise. From the labelled graph Γ , we construct a graph Γ' by removing every edge that is contained in a simple cycle and that is not a piece. We also remove the letters labelling those edges from the alphabet S to obtain the alphabet S' . Then the group defined by Γ' over the alphabet S' is isomorphic to $G(\Gamma)$ since the changes we made correspond to Tietze transformations. If Γ' is a forest, then the group defined by Γ is the free group on S' . Assume this is not the case. Then Γ' has girth at least 7.

Assume $G(\Gamma)$ is cyclic and hence abelian, and let $a, b \in S'$ with $a \neq b$. Then any diagram for the word $aba^{-1}b^{-1}$ over Γ' has more than one face. Since there are only 4 boundary edges and the girth of Γ' is at least 7, any face has at least 3 interior arcs. Therefore, the first curvature formula is violated.

We deduce that $G(\Gamma)$ is infinite cyclic if and only if $|S'| = 1$. Note that in this case, Γ' is a forest, and the free rank of the free product of the fundamental groups of the connected components of Γ is $|S| - 1$.

Proof of the theorem in the finitely presented case. If Γ has finitely many distinct simple cycles, Theorem 2.16 and Proposition 2.18 imply that the group is hyperbolic and torsion-free. A result of Koubi [Ko98, Theorem 5.1] implies that non-elementary hyperbolic groups

contain non-abelian free subgroups. Any torsion-free elementary hyperbolic group is either trivial or infinite cyclic. Therefore, the claim holds in this case. \square

We now make preliminary observations and definitions for the proof of the theorem in the remaining case. Let Γ be a labelled graph. Suppose there is an edge e that is not a piece. Then the label s of e appears only once on the graph. If e lies on a simple cycle, then removing e from Γ as well as s from S corresponds to a Tietze transformation and yields an isomorphic group. If e does not lie on a simple cycle, consider the presentation $\langle S | R \rangle$ for $G(\Gamma)$, where R is the set of words read on simple cycles in Γ . Then s appears in no word of R and therefore generates a free factor. In this case, the claim of the theorem holds. Therefore, we will assume that all edges in Γ are pieces throughout the rest of this section.

Assume that all edges on the labelled graph Γ are pieces. Let x, y be vertices lying in a connected component of Γ . The *piece distance* $d_p(x, y)$ is the least number of pieces whose concatenation is a path from x to y . We observe that the restriction of d_p to a connected component of Γ is a metric.

Lemma 3.4. *Let γ be a simple cycle on a $C(n)$ -labelled graph Γ . Let x be a vertex on γ . If n is even, there exists a vertex y on γ such that $d_p(x, y) = \frac{n}{2}$. If n is odd, one of the following holds:*

- *There is a vertex y on γ such that $d_p(x, y) = \frac{n+1}{2}$, or*
- *there are vertices y and z on γ such that $d_p(x, y) = d_p(x, z) = \frac{n-1}{2}$ and $d(y, z) = 1$. Any path starting at x and traversing both y and z consists of at least $\frac{n+1}{2}$ pieces. We call the edge connecting y and z the opposite edge to x .*

Proof. We observe: For any x, y in a connected component of Γ with $d_p(x, y) = d < \frac{n}{2}$, there is a unique simple path from x to y consisting of d pieces.

Now assume that the simple cycle γ is based at x . Let γ^+ be the longest initial subpath that can be made up of less than $\frac{n}{2}$ pieces, and let γ^- be the longest terminal subpath with this property. Then, by our observation, γ^+ and γ^- have only the vertex x in common. We may write $\gamma = \gamma^+ \delta \gamma^-$ for some nonempty simple path δ .

Assume that n is even. Then δ consists of at least 2 pieces and hence at least two edges. Any interior vertex y of δ satisfies $d_p(x, y) \geq \frac{n}{2}$.

Assume that n is odd. If δ consists of more than one edge, the argument above applies, and the first claim holds. If δ consists of exactly one edge, the second claim holds. \square

We now recall a basic fact about fundamental groups of graphs: Let Γ be a connected graph and T a spanning tree of Γ . Let E denote the set of edges in Γ but not in T . Then there is a bijective map from E to a free generating set of $\pi_1(\Gamma, \nu)$ (ν arbitrary in Γ). In particular, $\pi_1(\Gamma, \nu)$ is not finitely generated if and only if E is infinite. (For a proof, see [KM02, Section 6].)

Lemma 3.5. *Let Γ be a graph of bounded vertex-degree containing infinitely many pairwise distinct simple cycles. Then Γ contains infinitely many pairwise vertex-disjoint simple cycles.*

Proof. First assume that Γ has infinitely many connected components that have nontrivial fundamental group. Then the claim is obvious. Therefore, we will assume that Γ is connected.

We prove by contradiction. Let C be a maximal set of pairwise disjoint simple cycles on Γ and assume that C is finite. Consider a spanning tree T of Γ . Then there are infinitely many edges in Γ that do not lie in T , and therefore, since Γ is of bounded degree, T is infinite. The forest $T - C$ (i. e. T with all vertices and edges of elements of C removed) has finitely many connected components. Moreover, there are infinitely many edges in Γ that do not meet a vertex in C and that do not lie in T . Thus, one of the following must occur:

- There exists a component K of $T - C$ such that there is an edge in Γ but not in T and disjoint from C that connects two vertices of K , or
- there exist two components K, K' of $T - C$ such that there are two edges in Γ but not in T and disjoint from C connecting vertices of K and K' .

In both cases, we obtain a simple cycle that is vertex-disjoint from all elements of C . This contradicts the maximality of C . \square

Proof of the theorem in the infinitely presented case. We restrict ourselves to the case where Γ contains infinitely many pairwise distinct simple cycles. Otherwise $G(\Gamma)$ is finitely presented, which has been remarked upon.

First assume there are two vertex-disjoint simple cycles γ_1, γ_2 such that x, x' are vertices in γ_1 and y, y' are vertices in γ_2 with $4 = d_p(x, x') = d_p(y, y')$. The method of proof below indicates how to prove that the word w_1 read on a path from x to x' and the word w_2 read on a path from y to y' generate a rank 2 free subgroup of $G(\Gamma)$.

Now assume the first assumption does not hold. We deduce from Lemmas 3.4 and 3.5 that there are infinitely many pairwise vertex-disjoint simple cycles on Γ on each of which the maximal piece distance

between two points is 3. Moreover, for any vertex in one of these cycles γ , there is an opposite edge on γ . Let x_a, x'_a, x_b, x'_b be vertices with opposite edges a, a', b, b' in pairwise disjoint cycles. Since we are considering an infinite set of cycles labelled with a finite set of letters, we may assume that $\ell(a) = \ell(a')$ and $\ell(b) = \ell(b')$.

We add α, β to the alphabet S and add the following relations to the presentation:

- $R_\alpha := \{\alpha \text{red}(\ell(p)^{-1}\ell(p')) | p : \iota(a) \rightarrow x_a, p' : \iota(a') \rightarrow x'_a\},$
- $R_\beta := \{\beta \text{red}(\ell(p)^{-1}\ell(p')) | p : \iota(b) \rightarrow x_b, p' : \iota(b') \rightarrow x'_b\},$

where $\text{red}(-)$ denotes the free reduction, $p : w \rightarrow x$ denotes a path from w to x , and $\iota(e)$ denotes the initial vertex of an oriented edge e . We call a face in a diagram bearing a relator in $R_\alpha \cup R_\beta$ *special*. Let R denote the set of labels of all cycles on Γ . Then

$$G(\Gamma) = \langle S | R \rangle \cong \langle S \cup \{\alpha, \beta\} | R \cup R_\alpha \cup R_\beta \rangle,$$

since we only applied the Tietze transformations of adding generators. (Note that $\ell(p_1) = \ell(p_2)$ in G for any two paths p_1, p_2 from vertices x to y .) We show that α and β generate a free subgroup of rank 2 in $G(\Gamma)$.

Let D be a nontrivial minimal diagram for a cyclically reduced word ω in α, β over the extended presentation. First, we can assume that no edge labelled α or β is an interior edge: If there is such an interior edge, it can only occur when two special faces are glued together in such a way that the boundary read around these two faces can be obtained by gluing together two cycles lifting to Γ . Thus we can replace the two faces containing α (or β) by two that do not.

Assume that no interior edge originates from Γ (if we associate to the interior paths of special faces their obvious lifts to Γ). Then interior arcs are pieces, and we are dealing with a $(3, 7)$ -diagram.

We construct from D a diagram D' that violates the first curvature formula by replacing each special face by two faces over the original presentation and performing some foldings. For the sake of brevity we describe the process for one special face F whose label lies in R_α :

The subpath of ∂F that lies in the interior of D comes from two reduced paths $p : \iota(a) \rightarrow x_a$ and $p' : \iota(a') \rightarrow x'_a$, where a free reduction may have occurred corresponding to cutting off initial subpaths of p and p' that bear the same labels. We replace F by two faces f and f' constructed as follows: $\partial f = \ell(p)\ell(q)$ and $\partial f' = \ell(p')\ell(q')$, where q (resp. q') is a path in Γ such that the concatenation pq (resp. $p'q'$) is a closed reduced path in Γ . The two faces are glued together at the vertex lifting to the initial vertex of p resp. p' , and edges of f and

f' are folded together starting at this vertex in correspondence to the free reduction of $\ell(p)^{-1}\ell(p')$. Then we can glue these two faces into the diagram instead of F . Now there are three cases to consider:

- (A) Suppose neither $\ell(p)$ nor $\ell(p')$ start with $\ell(a)$. Then we can choose q and q' such that $\ell(q)$ and $\ell(q')$ terminate with $\ell(a)^{-1}$. Thus we can fold f and f' together along the corresponding edge as well. Then the interior paths of f and f' consist of at least 4 pieces by Lemma 3.4.
- (B) Suppose both start with a . Then p and p' consist of at least 4 pieces by Lemma 3.4.
- (C) Suppose $\ell(p)$ starts with $\ell(a)$, and $\ell(p')$ does not (or the symmetric case). Then f has interior degree at least 4, f' has interior degree at least 3, and there is no folding between f and f' . We can, however just fold together the two exterior edges of f and f' incident at the vertex where the two faces intersect (disregarding the label) to increase the interior degrees of f and f' . The resulting diagram is still a $(3, 7)$ -diagram.

For all cases we obtain that the constructed boundary faces f and f' have interior degree at least 4. If we perform this procedure for all special faces, we obtain a $(3, 7)$ -diagram violating the first curvature formula, which is a contradiction.

We are left with the case that D does contain interior edges originating from Γ . As above, we can construct from D a diagram D' over Γ (in particular making the choices in case (A)), but without performing the foldings in case (A) or case (C).

Our observations from Section 2 yield that since D is minimal, any edge lying on an interior face does not originate from Γ . Thus an interior edge e originating from Γ lies in the boundary of one or two special faces, and therefore it lies in the intersection of two boundary faces f_1 and f_2 of D' . We claim: Removing $f_1 \cup f_2$ from D' separates D' into two nontrivial diagrams.

Since the boundary word of D is cyclically reduced, there can be no vertex in $\partial D' \cap f_1 \cap f_2$ whose lifts via f_1 and f_2 coincide. (This follows from the fact that the vertices x_a, x'_a, x_b, x'_b were chosen on vertex-disjoint cycles.) If $\partial D' \cap f_1 \cap f_2$ were to be non-empty, f_1 and f_2 would enclose some nontrivial subdiagram made up of interior faces. This would be a nontrivial $[3, 6]$ -diagram with at most one boundary vertex of degree less than 3, contradicting the second curvature formula.

Thus $\partial D' \cap (f_1 \cup f_2)$ is not contiguous, and on both sides of $f_1 \cup f_2$ there are nontrivial subdiagrams. We call two faces of D' containing an edge originating from Γ a *bottle neck*.

If there exists a bottle neck, there also exists an *extremal* bottle neck $f_1 \cup f_2$, i.e. one such that in one component C of $D' \setminus (f_1 \cup f_2)$ there are no more bottle necks. This means in the diagram $\Delta := C \cup f_1 \cup f_2$, all interior edges originating from Γ lie in $f_1 \cap f_2$.

Now consider the diagram Δ . We can remove all edges originating from Γ to merge f_1 and f_2 to one face f . The resulting diagram has no more interior edges originating from Γ . Thus we can apply our technique from above: Performing the foldings indicated in case (A) and case (C) above yields a $(3, 7)$ -diagram where all but one boundary face (namely f) have interior degree at least 4. This contradicts the first curvature formula. \square

4. COARSE EMBEDDING OF THE GRAPH

We now consider the question of how Γ embeds into $G(\Gamma)$. Similar arguments have been used by Ollivier [Ol06] to show that a connected $C'(\frac{1}{6})$ -labelled graph embeds isometrically into $G(\Gamma)$.

Let Γ be a connected, labelled graph. Let v be a vertex in Γ and $g \in G(\Gamma)$. There is a unique map of labelled graphs $\iota : \Gamma \rightarrow \text{Cay}(G(\Gamma), S)$ satisfying $\iota(v) = g$. We prove a claim of Gromov [Gr03, Theorem 2.3]:

Lemma 4.1. *Let Γ be a connected component of a $Gr(6)$ -labelled graph. Then the map $\iota : \Gamma \rightarrow \text{Cay}(G(\Gamma), S)$ is injective.*

Proof. Let $x \neq y \in \Gamma$, and assume $\iota(x) = \iota(y)$. Let p be a path from x to y on Γ such that the area of $w = \ell(p)$ is minimal among all paths from x to y . We may assume that w is reduced. Since $x \neq y$, this area is greater than zero. Let D be a minimal diagram for w over Γ . For each boundary edge of D bounding a face, there is a lift to Γ via p and one via the face it lies on. Suppose there were an edge e bounding a face f for which these two lifts were essentially equal. Then we could remove e and the interior of f . The resulting diagram would have a boundary word read on a path connecting x and y of lesser area than D , which is a contradiction. Thus every arc of D bounding a face is a piece. By construction, D has at most one spur (connecting the base vertex to the rest of the diagram) and hence at most one vertex of degree one. Now forgetting vertices of degree two leaves a $[3, 6]$ -diagram violating the second curvature formula, which is a contradiction. \square

Corollary 4.2. *For any $C(6)$ -labelled graph Γ with a connected component having more than one vertex, the group $G(\Gamma)$ is infinite.*

Proof. This follows since $G(\Gamma)$ is nontrivial and, by Proposition 2.18, torsion-free. \square

Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of connected finite graphs and let $\Gamma := \bigsqcup_{n \in \mathbb{N}} \Gamma_n$ be their disjoint union. We endow Γ with a metric that coincides with the graph metric on each connected component such that $d(\Gamma_{a_n}, \Gamma_{b_n}) \rightarrow \infty$ as $a_n + b_n \rightarrow \infty$ assuming $a_n \neq b_n$ for almost all n . (For example, set $d(x, y) = \text{diam}(X_m) + \text{diam}(X_n) + m + n$ if $x \in X_m$, $y \in X_n$ and $m \neq n$.) We call the resulting metric space the *coarse union* of the Γ_n .

If $\text{Cay}(G(\Gamma), S)$ is infinite, it contains an infinite geodesic ray. Then we can map Γ into $\text{Cay}(G(\Gamma), S)$ via a map of labelled graphs ι by lining the Γ_n 's up on this geodesic ray such that, for all sequences a_n, b_n we have $d(\iota(\Gamma_{a_n}), \iota(\Gamma_{b_n})) \rightarrow \infty$ if and only if $d(\Gamma_{a_n}, \Gamma_{b_n}) \rightarrow \infty$. We claim that such a map ι is a *coarse embedding*, i.e. it satisfies for every sequence of pairs of points $(x_n, y_n)_{n \in \mathbb{N}}$ in $\Gamma \times \Gamma$:

$$d(x_n, y_n) \rightarrow \infty \Leftrightarrow d(\iota(x_n), \iota(y_n)) \rightarrow \infty.$$

Theorem 4.3. *Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of connected finite graphs such that $\Gamma := \bigsqcup_{n \in \mathbb{N}} \Gamma_n$ is $\text{Gr}(6)$ -labelled and such that $|\Gamma_n|$ is unbounded. Then the coarse union Γ embeds coarsely into $\text{Cay}(G(\Gamma), S)$.*

Proof. First note that the assumption implies that $\text{Cay}(G(\Gamma), S)$ is infinite, since by Lemma 4.1, each Γ_n maps injectively into $\text{Cay}(G, S)$.

Let $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence of pairs of points such that $d(x_n, y_n) \rightarrow \infty$. We claim: $d(\iota(x_n), \iota(y_n)) \rightarrow \infty$. By the above construction of ι , it is sufficient to consider the case where for all n , x_n and y_n lie in the same connected component. Suppose our claim is false. Then (x_n, y_n) has a subsequence (x'_n, y'_n) such that $d(\iota(x'_n), \iota(y'_n))$ is bounded. Hence there is a subsequence (x''_n, y''_n) such that for all n , the labels of paths from x''_n to y''_n define the same element w of G . Since all Γ_n are bounded, we also assume that for $n \neq m$, the graphs containing $\{x''_n, y''_n\}$ and $\{x''_m, y''_m\}$ are distinct and non-isomorphic.

Let $n \in \mathbb{N}$ and choose paths p_n, q_n from x''_1 to y''_1 , respectively from x''_n to y''_n , such that the minimal area of a diagram for the word $\ell(p_n)\ell(q_n)^{-1}$ over Γ is minimal. We claim that the area of such a minimal diagram D_n is zero. We may assume that the paths are reduced. We can argue as in Lemma 4.1 that every arc bounding a face in D_n is a piece. Now forgetting vertices of degree two yields a $[3, 6]$ -diagram with at most two vertices of degree one. By the second curvature formula, such a diagram has exactly two vertices and hence no faces. Thus D_n has no face, $\ell(p_n) = \ell(q_n)$, and p_n and q_n are essential pieces and hence simple paths.

Let X be the connected component of Γ containing x''_1 . Then $|q_n| = |p_n|$ is bounded from above by the maximal length of a simple path

on the finite graph X . Since n was arbitrary, this contradicts the assumption $d(x''_n, y''_n) \rightarrow \infty$.

Obviously, $d(\iota(x), \iota(y)) \leq d(x, y)$ for any x, y in a connected component of Γ . Hence $d(\iota(x_n), \iota(y_n)) \rightarrow \infty$ implies $d(x_n, y_n) \rightarrow \infty$. \square

Example 4.4. Let $p \in \mathbb{N}$. We construct a sequence of finite, connected labelled graphs $(\Gamma_n)_{n \in \mathbb{N}}$ such that their disjoint union Γ has a $C(p)$ -labelling and such that any map of labelled graphs $\iota : \Gamma \rightarrow \text{Cay}(G(\Gamma), S)$ is *not* a quasi-isometric embedding. Let $S = S_1 \cup S_2 \cup \{a, b\}$, where S_1, S_2 and $\{a, b\}$ are pairwise disjoint and $|S_1| > 1$. Let $(w_n)_{n \in \mathbb{N}}$ be a sequence of pairwise distinct, reduced words in the free monoid $M(S_1)$ on $S_1 \cup S_1^{-1}$ such that $|w_n| = O(\log n)$, and let $(v_n)_{n \in \mathbb{N}}$ be a sequence of pairwise distinct, reduced words in $M(S_2)$. For each n , let Γ_n be the graph given in Figure 5:

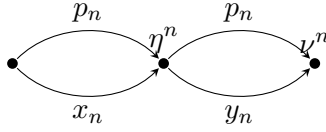


FIGURE 5. The graph Γ_n .

Here p_n, x_n, y_n are words in S given as follows:

- $p_n = b^{f(n)}$, where $f(n)$ is some function $\mathbb{N} \rightarrow \mathbb{N}$ such that $\log n = o(f(n))$,
- $x_n = w_{(p-1)n+1} a w_{(p-1)n+2} a \dots a w_{pn} a$,
- $y_n = v_{(p-1)n+1} a v_{(p-1)n+2} a \dots a v_{pn} a^{f(n)}$.

The labels η^n and ν^n are vertex-labels for reference. The reader can easily check that the resulting graph Γ has a $C(p)$ -labelling by considering how many non-consecutive instances of a can occur in a piece. By construction $|p_n| \leq |y_n|$ and hence $d(\eta^n, \nu^n) = |p_n|$. Since p_n and x_n are equal in $G(\Gamma)$, we have $d(\iota(\eta^n), \iota(\nu^n)) \leq |x_n| = o(d(\eta^n, \nu^n))$. Therefore, ι cannot be a quasi-isometric embedding.

5. LACUNARY HYPERBOLICITY

We now consider an infinite sequence of finite graphs $(\Gamma_n)_{n \in \mathbb{N}}$ such that their disjoint union is $Gr(7)$ -labelled or $Gr'(\frac{1}{6})$ -labelled. The resulting group is a limit of hyperbolic groups. In the view of [OOS09] it is natural to ask whether these groups are *lacunary hyperbolic*. We give some preliminary definitions:

Definition 5.1 (Ultrafilter). An *ultrafilter* is a finitely additive map $\omega : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ such that $\omega(\mathbb{N}) = 1$. An ultrafilter ω is called *non-principal* if $\omega(F) = 0$ for all finite subsets F of \mathbb{N} . Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a sequence. Then for $x \in \mathbb{R}$ we say $x = \lim_{\omega} f(n)$ if $\forall \epsilon > 0 : \omega(f^{-1}([x - \epsilon, x + \epsilon])) = 1$.

It is a fact that given an ultrafilter ω , any bounded sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ has a limit with respect to ω .

Definition 5.2 (Asymptotic cone). Let ω be a non-principal ultrafilter and $(d_n)_{n \in \mathbb{N}}$ a sequence of real numbers such that $d_n \rightarrow \infty$ as $n \rightarrow \infty$. The sequence d_n is called *scaling sequence*. Let G be a group generated by a finite set S . Let $G^{\mathbb{N}}$ denote the space of sequences of elements of G , and $X := \{(x_n) \in G^{\mathbb{N}} \mid d_S(1, x_n) = O(d_n)\}$, where d_S denotes the distance in $\text{Cay}(G, S)$. We define a pseudo-metric d on X by setting $d((x_n), (y_n)) = \lim_{\omega} \frac{d(x_n, y_n)}{d_n}$. An equivalence relation on X is induced by $(x_n) \sim (y_n) :\Leftrightarrow d(x_n, y_n) = 0$. The *asymptotic cone* of G with respect to S , ω and (d_n) is defined as X/\sim with the metric induced by d . We denote it by $\text{Con}^w(G, d_n)$.

Let G be a group generated by a finite set S , and let $\pi : G \rightarrow H$ be a group homomorphism. Then the *injectivity radius* of π , denoted $r_S(\pi)$, is the largest $r \in \mathbb{R}$ such that π restricted to the open ball of radius r at 1 in $\text{Cay}(G, S)$ is injective. An \mathbb{R} -tree is a 0-hyperbolic space (see Definition 2.14). In particular, an \mathbb{R} -tree contains no nontrivial embedded cycles. We recall the definition and a characterization of lacunary hyperbolicity from [OOS09, Section 3.1]:

Definition 5.3 (Lacunary hyperbolic group). A finitely generated group G is called *lacunary hyperbolic* if the following equivalent conditions hold:

- Some asymptotic cone of G is an \mathbb{R} -tree.
- G is the direct limit of finitely generated groups and epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots$$

such that G_i is generated by a finite set S_i , $\alpha_i(S_i) = S_{i+1}$ and each G_i is δ_i -hyperbolic, where $\delta_i = o(r_{S_i}(\alpha_i))$.

5.1. The $Gr(7)$ case. Since the defining condition for lacunary hyperbolicity has a metric nature, the $Gr(7)$ condition does not yield optimal results. We are able to prove, however:

Proposition 5.4. *Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of finite, connected labelled graphs such that their disjoint union is $Gr(7)$ -labelled. Then there*

exists an infinite subsequence of graphs $(\Gamma_{k_n})_{n \in \mathbb{N}}$ such that $G(\bigsqcup_{n \in \mathbb{N}} \Gamma_{k_n})$ is lacunary hyperbolic.

Lemma 5.5. *Let Γ be a finite labelled graph and $(\Gamma_n)_{n \in \mathbb{N}}$ a sequence of connected, finite labelled graphs such that $\Gamma' := \Gamma \sqcup \bigsqcup_{n \in \mathbb{N}} \Gamma_n$ is $Gr(7)$ -labelled and such that for $n \neq n'$, the labelled graphs Γ_n and $\Gamma_{n'}$ are non-isomorphic. Then the injectivity radii ρ_n of the projections $\pi_n : G(\Gamma) \rightarrow G(\Gamma \sqcup \Gamma_n)$ induced by the identity on S tend to infinity.*

Proof. Suppose this is false. Then there exists an infinite sequence $(k_n)_{n \in \mathbb{N}}$ and a word w such that w is nontrivial in $G(\Gamma)$ and such that for all $n \in \mathbb{N}$, w is trivial in $G(\Gamma \sqcup \Gamma_{k_n})$ and $|w| = \rho_{k_n}$. For each $n \in \mathbb{N}$, let D_n be a diagram for w over $\Gamma \sqcup \Gamma_{k_n}$ such that the labels of all faces are read on simple cycles and such that no interior edge originates from the graph. By construction, each D_n contains at least one face from Γ_{k_n} . For each $n \in \mathbb{N}$, let D'_n be an inclusion-minimal subdiagram of D_n containing all faces that lift to Γ_{k_n} . Then all boundary faces of D'_n lift to Γ_{k_n} .

The first curvature formula implies that for each n , there is at least one boundary face f_n of D'_n such that one arc p_n of $f_n \cap \partial D'_n$ is not an essential piece in Γ' . This implies that all p_n bear distinct labels and hence $|p_n| \rightarrow \infty$. Note that p_n is a subpath of ∂D_n and/or the boundaries of some faces of D_n that lift to Γ . By Theorem 2.16, the length of any such path is bounded by $|w| + 8|w|(V(\Gamma) + 1)$, where $V(\Gamma)$ is the number of vertices of Γ (Hence $V(\Gamma) + 1$ is an upper bound for the length of a simple closed path on Γ). This is a contradiction. \square

Proof of the proposition. If all but finitely many Γ_n are isomorphic, we can consider $G(\Gamma)$ as given by a finite $Gr(7)$ -labelled graph. In that case, $G(\Gamma)$ is hyperbolic and hence lacunary hyperbolic.

Therefore, we can assume that for $n \neq n'$, Γ_n and $\Gamma_{n'}$ are non-isomorphic. We choose the subsequence recursively: Let $k_0 = 1$, and let k_1, \dots, k_N be chosen. Set $\Gamma^N := \bigsqcup_{i=1}^N \Gamma_{k_i}$. Then, by Theorem 2.16, $G(\Gamma^N)$ is δ_N -hyperbolic for some $\delta_N > 0$. By Lemma 5.5, the injectivity radii ρ_n of the maps $G(\Gamma^N) \rightarrow G(\Gamma^N \sqcup \Gamma_n)$ induced by the identity on S tend to infinity. Hence we may choose k_{N+1} such that $k_{N+1} > \max\{k_1, \dots, k_N\}$ and such that $\rho_{k_{N+1}} > N\delta_N$. The resulting limit group $G(\bigsqcup_{i \in \mathbb{N}} \Gamma_{k_i})$ is lacunary hyperbolic. \square

5.2. The $Gr'(\frac{1}{6})$ case. Finitely presented $C'(\frac{1}{6})$ groups have been investigated in [Oll]. (In fact, Ollivier uses a stronger variant of the $C'(\frac{1}{6})$ condition.) His proofs of the facts we will use generalize to our $Gr'(\frac{1}{6})$

condition. The question, when classical $C'(\frac{1}{6})$ groups are lacunary hyperbolic has been solved in [OOS09, Proposition 3.12]. We follow their arguments to extend this result:

Proposition 5.6. *Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of finite, connected graphs such that*

- $\Gamma := \sqcup_{n \in \mathbb{N}} \Gamma_n$ has a $Gr'(\frac{1}{6})$ -labelling and
- $\Delta(\Gamma_n) = O(g(\Gamma_n))$,

where $\Delta(\Gamma_n)$ denotes the diameter and $g(\Gamma_n)$ denotes the girth of each graph. (If Γ_n is a tree, set $g(\Gamma_n) = 0$.) Then $G(\Gamma)$ is lacunary hyperbolic if and only if the set of girths $L := \{g(\Gamma_n) | n \in \mathbb{N}\}$ is sparse, i.e. for all $K \in \mathbb{R}^+$ there exists $a \in \mathbb{R}^+$ such that $[a, aK] \cap L = \emptyset$.

We first give a precise estimate for the hyperbolicity constant of a group given by a finite $Gr'(\frac{1}{6})$ -labelled graph. We observe as in Section 2:

Lemma 5.7. *Let Γ be a $Gr'(\frac{1}{6})$ -labelled graph. Let $w \in M(S)$ satisfying $w = 1$ in $G(\Gamma)$ and let D be a minimal diagram over Γ for w . Let f be a face of D . Then f is simply connected. Any interior arc of ∂f has length less than $\frac{1}{6}|\partial f|$.*

Note that for a singular disk diagram D over a group presentation $G = \langle S | R \rangle$, the 1-skeleton of D maps into $\text{Cay}(G, S)$. Singular disk diagrams with the property of Lemma 5.7 whose boundaries are geodesic triangles in the Cayley graph have been classified in [St90, Section 3.4]. This classification yields:

Lemma 5.8. *Let Γ be a finite $Gr'(\frac{1}{6})$ -labelled graph, and let Δ be the maximum of the diameters of its connected components. Then $G(\Gamma)$ is 2Δ -hyperbolic.*

We will use the following two facts [Ol06, Lemma 13 and Theorem 1] proven by Ollivier:

Lemma 5.9. *Let Γ be a $Gr'(\frac{1}{6})$ -labelled graph. Let $w \in M(S)$ satisfying $w = 1$ in $G(\Gamma)$, and let D be a minimal diagram for w over Γ . Then for any face f , we have $|\partial f| \leq |\partial D|$.*

Theorem 5.10. *Let Γ be a $Gr'(\frac{1}{6})$ -labelled graph. Then each connected component of Γ embeds isometrically into $\text{Cay}(G(\Gamma), S)$.*

Proof of the proposition. Note that given the previous lemmas and theorem, a proof can be deduced from [OOS09, Proof of Proposition 3.12]. In fact, the first part of our proof uses the same arguments.

By identifying isomorphic connected components of Γ , we can assume that for $n \neq n'$, Γ_n and $\Gamma_{n'}$ are non-isomorphic. Assume that L is sparse. Then there exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of positive real numbers such that for all $n \in \mathbb{N}$, we have $L \cap [\alpha_n, n\alpha_n] = \emptyset$. We may assume $n\alpha_n < \alpha_{n+1}$ for all n . Fix $C \in \mathbb{R}^+$ such that for all n : $\Delta(\Gamma_n) \leq Cg(\Gamma_n)$. Let

$$\Gamma^k := \sqcup_{g(\Gamma_n) < \alpha_k} \Gamma_n.$$

Then for all k , the graph Γ^k is a finite and $Gr'(\frac{1}{6})$ -labelled. For any connected component of Γ^k , the girth is bounded from above by α_k , and hence the diameter is bounded by $C\alpha_k$. Lemma 5.8 implies that $G_k := G(\Gamma^k)$ is $2C\alpha_k$ -hyperbolic. Set $\delta_k := 2C\alpha_k$.

The injectivity radius $r_S(\pi_k)$ of the map $G_k \rightarrow G_{k+1}$ induced by the identity on S is the length of the shortest word w in S that is trivial in G_{k+1} but not trivial in G_k . Hence a minimal diagram D for w over Γ^{k+1} has a face f from $\Gamma^{k+1} \setminus \Gamma^k$. Such a face satisfies $|\partial f| \geq k\alpha_k$ by construction. Hence Lemma 5.9 implies $r_S(\pi_k) = |w| \geq k\alpha_k$. Therefore $\frac{\delta_k}{r_S(\pi_k)} \rightarrow 0$ as $k \rightarrow \infty$, and $G(\Gamma)$ is lacunary hyperbolic.

We now prove the converse. Assume that the set of girths L is not sparse. Choose any non-decreasing scaling sequence $(d_n)_{n \in \mathbb{N}}$ tending to infinity and any non-principal ultrafilter ω on \mathbb{N} . We show that $Y := \text{Con}^\omega(G(\Gamma), d_n)$ is not an \mathbb{R} -tree. Theorem 5.10 implies that from each graph Γ_n , there is a cycle p_n of length $g(\Gamma_n)$ isometrically embedded into $X := \text{Cay}(G(\Gamma), S)$. Since L is not sparse, there exists $K > 0$ such that for all $a > 0$ we have $[a, aK] \cap L \neq \emptyset$. Hence for all $n \in \mathbb{N}$ there is $k(n) \in \mathbb{N}$ such that the inequality

$$d_n \leq g(\Gamma_{k(n)}) \leq Kd_n$$

holds, or in other words $1 \leq |p_{k(n)}|/d_n \leq K$. Since the interval $[1, K]$ is bounded, the sequence $(|p_{k(n)}|/d_n)_{n \in \mathbb{N}}$ converges to some $R \in [1, K]$ with respect to the ultrafilter ω . We may view each cycle $p_{k(n)}$ as a continuous map $p_{k(n)} : \mathbb{R}/\mathbb{Z} \rightarrow X$, and we may assume that for all t and for all $\epsilon \in (-\frac{1}{2}, \frac{1}{2}]$ we have $d(p_{k(n)}(t), p_{k(n)}(t + \epsilon)) = |\epsilon||p_{k(n)}|$.

Consider the cycle $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow Y, t \mapsto [(p_{k(n)}(t))_{n \in \mathbb{N}}]$. Let $t \neq t' \in \mathbb{R}/\mathbb{Z}$ and $\epsilon \in (-\frac{1}{2}, \frac{1}{2}]$ such that $t' = t + \epsilon$. Then for any n ,

$$d(\gamma(t), \gamma(t')) = \lim_n^\omega \frac{d(p_{k(n)}(t), p_{k(n)}(t + \epsilon))}{d_n} = \lim_n^\omega |\epsilon| \frac{|p_{k(n)}|}{d_n} = |\epsilon|R > 0.$$

Hence γ is injective, and thus Y is not an \mathbb{R} -tree. \square

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